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Solvability for a coupled system of nonlinear fractional integro-differential equations

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Abstract. In this paper, we prove the existence of solutions for fractional coupled systems of integro-differential equations in Banach spaces. The results are obtained using fractional calculus and fixed point theorems. Some concrete examples are also presented to illustrate the possible application of the established analytical results.

Keywords: Caputo derivative, fixed point theorem, coupled system, existence, uniqueness.

MSC 2000 classification: primary 34A34, secondary 34B10.

1 Introduction

The fractional differential equations theory is increasingly used for many mathematical models in science and engineering, such as physics, chemistry, control theory, signal processing and biophysics. For more details, we refer the reader to ([7, 9, 11]) and the references therein. Recently, there has been a significant progress in the investigation of these equations (see [2, 4, 5]). Moreover, the study of coupled systems of fractional differential equations is also of great importance. Such systems occur in various problems of applied science. For some recent results on the fractional systems, we refer the reader to (see [1, 3, 6, 12]).

In this paper, we study the following coupled system of fractional integro-

differential equations:

$$\left\{ \begin{array}{l} D^\alpha u(t) = f_1(t, u(t), v(t)) + \int_0^t \frac{(t-s)^{\sigma-1}}{\Gamma(\sigma)} g_1(s, u(s), v(s)) ds, t \in [0, 1], \\ D^\beta v(t) = f_2(t, u(t), v(t)) + \int_0^t \frac{(t-s)^{\theta-1}}{\Gamma(\theta)} g_2(s, u(s), v(s)) ds, t \in [0, 1], \\ \sum_{k=0}^{n-2} (|u^{(k)}(0)| + |v^{(k)}(0)|) = 0, \\ u^{(n-1)}(0) = \gamma I^p u(\eta), \eta \in]0, 1[, \\ v^{(n-1)}(0) = \delta I^q v(\zeta), \zeta \in]0, 1[, \end{array} \right. \quad (1)$$

where D^α and D^β denote the Caputo fractional derivatives with $n-1 < \alpha < n$, $n-1 < \beta < n$, $n \in \mathbb{N}^*$, $\delta, \gamma \in \mathbb{R}^*$, σ, θ, p and q are non negative reals numbers, f_1, f_2, g_1 and g_2 are the functions which will be specified later.

To the best of our knowledge, no paper has considered the fractional integro-differential system (1) with two arbitrary orders α, β and using Riemann-Liouville fractional integral conditions. Our aim is to study the existence and uniqueness of solutions of the fractional coupled system (1). However, with fractional integral conditions, it will become more complicated. So, we can take the first $n-1$ classical derivatives equal to 0, for the unknown functions u and v . Then, we shall use some fixed point theorems and other technics to overcome the difficulties.

We organize the rest of this paper as follows: In section 2, we present some definitions, preliminaries and lemmas. In Section 3, we prove our main results for the problem (1). At the last section, some examples are presented to illustrate the effectiveness of the main results.

2 Preliminaries

The following notations, definitions and preliminary facts will be used throughout this paper.

Definition 1. The Riemann-Liouville fractional integral operator of order $\alpha > 0$, for a continuous function f on $[a, b]$ is defined as:

$$I^\alpha f(t) = \int_a^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} f(\tau) d\tau, \alpha > 0, a < t < b, \quad (2)$$

where $\Gamma(\alpha) := \int_0^\infty e^{-u} u^{\alpha-1} du$.

Definition 2. The Caputo fractional derivative of order $\alpha > 0$ for $f \in C^n([a, b], \mathbb{R})$ is defined as:

$$D^\alpha f(t) = \int_a^t \frac{(t-\tau)^{n-\alpha-1}}{\Gamma(n-\alpha)} f^{(n)}(\tau) d\tau, n-1 < \alpha < n, n \in \mathbb{N}^*, a < t < b. \quad (3)$$

For more details about fractional calculus, see [10].

The following lemmas give some properties of Riemann-Liouville fractional integrals and Caputo fractional derivatives [7, 9].

Lemma 1. *Let $r, s > 0$, $f \in L^1([a, b])$. Then $I^r I^s f(t) = I^{r+s} f(t)$, $D^s I^s f(t) = f(t)$, $t \in [a, b]$.*

Lemma 2. *Let $s > r > 0$, $f \in L^1([a, b])$. Then $D^r I^s f(t) = I^{s-r} f(t)$, $t \in [a, b]$.*

To study the coupled system (1), we need the following two lemmas [8, 10].

Lemma 3. *For $\alpha > 0$, the general solution of the equation $D^\alpha x(t) = 0$ is given by*

$$x(t) = c_0 + c_1 t + c_2 t^2 + \dots + c_{n-1} t^{n-1}, \quad (4)$$

where $c_i \in \mathbb{R}$, $i = 0, 1, 2, \dots, n-1$, $n = [\alpha] + 1$.

Lemma 4. *Let $\alpha > 0$. Then*

$$I^\alpha D^\alpha x(t) = x(t) + c_0 + c_1 t + c_2 t^2 + \dots + c_{n-1} t^{n-1}, \quad (5)$$

for some $c_i \in \mathbb{R}$, $i = 0, 1, 2, \dots, n-1$, $n = [\alpha] + 1$.

We prove the following auxiliary result:

Lemma 5. *Let $g \in C([0, 1], \mathbb{R})$, $n-1 < \alpha < n$, $\sigma > 0$, $n \in \mathbb{N}^*$. Then, the solution of the problem*

$$D^\alpha x(t) = f(t) + \int_0^t \frac{(t-s)^{\sigma-1}}{\Gamma(\sigma)} g(s) ds, \quad (6)$$

associated with the conditions

$$|x(0)| + |x'(0)| + \dots + |x^{(n-2)}(0)| = 0$$

and

$$x^{(n-1)}(0) = \gamma I^p x(\eta), \eta \in]0, 1[, p > 0,$$

is given by

$$\begin{aligned}
 x(t) = & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s) ds + \int_0^t \frac{(t-s)^{\alpha+\sigma-1}}{\Gamma(\alpha+\sigma)} g(s) ds \\
 & + \frac{\gamma \Gamma(p+n) t^{n-1}}{\Gamma(n) (\Gamma(p+n) - \gamma \eta^{p+n-1})} \\
 & \times \left(\int_0^\eta \frac{(\eta-s)^{p+\alpha-1}}{\Gamma(p+\alpha)} f(s) ds + \int_0^\eta \frac{(\eta-s)^{p+\alpha+\sigma-1}}{\Gamma(p+\alpha+\sigma)} g(s) ds \right), \quad (7)
 \end{aligned}$$

provided that $\gamma \neq \frac{\Gamma(p+n)}{\eta^{p+n-1}}$.

Proof: We have

$$\begin{aligned}
 x(t) = & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s) ds + \int_0^t \frac{(t-\tau)^{\alpha+\sigma-1}}{\Gamma(\alpha+\sigma)} g(\tau) d\tau \\
 & - c_0 - c_1 t - c_2 t^2 - \dots - c_{n-1} t^{n-1}. \quad (8)
 \end{aligned}$$

For $k = 1, 2, \dots, n-1$, we obtain

$$x^{(k)}(0) = -k! c_k.$$

Since $x(0) = x'(0) = \dots = x^{(n-2)}(0) = 0$, then $c_0 = c_1 = \dots = c_{n-2} = 0$.

Thanks to Lemma 2, we can write

$$\begin{aligned}
 I^p x(t) = & \int_0^t \frac{(t-s)^{p+\alpha-1}}{\Gamma(p+\alpha)} f(s) ds \\
 & + \int_0^t \frac{(t-s)^{p+\alpha+\sigma-1}}{\Gamma(p+\alpha+\sigma)} g(s) ds - c_{n-1} \frac{\Gamma(n) t^{p+n-1}}{\Gamma(p+n)}. \quad (9)
 \end{aligned}$$

Using the last boundary condition of Lemma 5, we get

$$c_{n-1} = -\frac{\gamma \Gamma(p+n)}{\Gamma(n) (\Gamma(p+n) - \gamma \eta^{p+n-1})} (I^{p+\alpha} f(\eta) + I^{p+\alpha+\sigma} g(\eta)). \quad (10)$$

Substituting $(c_i)_{i=0, \dots, n-1}$ in (8), we obtain the desired quantity (7).

Let us now introduce the spaces:

$$X := \{u(t) / u \in C^n([0, 1], \mathbb{R}^+)\},$$

$$Y := \{v(t) / v \in C^n([0, 1], \mathbb{R}^+)\}.$$

We define the norms

$$\|u\|_X := \sup_{t \in [0,1]} |u(t)|, \|v\|_Y := \sup_{t \in [0,1]} |v(t)|.$$

For the Banach space $X \times Y$, we define the norm

$$\|(u, v)\|_{X \times Y} := \max \left(\|u\|_X, \|v\|_Y \right).$$

3 Main Results

We introduce the following quantities:

$$\begin{aligned} \omega_1 &= \frac{|\gamma| \Gamma(p+n)}{\Gamma(n) (\Gamma(p+n) - \gamma \eta^{p+n-1})}, \\ \omega_2 &= \frac{|\delta| \Gamma(q+n)}{\Gamma(n) (\Gamma(q+n) - \delta \zeta^{q+n-1})}, \end{aligned}$$

$$\begin{aligned} M_1 &: = \frac{1}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha+\sigma+1)} + \frac{|\omega_1| \eta^{p+\alpha}}{\Gamma(p+\alpha+1)} + \frac{|\omega_1| \eta^{p+\sigma+\alpha}}{\Gamma(p+\alpha+\sigma+1)}, \\ M_2 &: = \frac{1}{\Gamma(\beta+1)} + \frac{1}{\Gamma(\beta+\theta+1)} + \frac{|\omega_2| \zeta^{q+\beta}}{\Gamma(q+\beta+1)} + \frac{|\omega_2| \zeta^{q+\theta+\beta}}{\Gamma(q+\theta+\beta+1)}. \end{aligned}$$

Also, we consider the following hypotheses

(H1) : There exist non negative real numbers $m_i, m'_i, n_i, n'_i, (i = 1, 2)$, such that for all $t \in [0, 1], (u_1, v_1), (u_2, v_2) \in \mathbb{R}^2$, we have

$$\begin{aligned} |f_1(t, u_2, v_2) - f_1(t, u_1, v_1)| &\leq m_1 |u_2 - u_1| + m_2 |v_2 - v_1|, \\ |g_1(t, u_2, v_2) - g_1(t, u_1, v_1)| &\leq m'_1 |u_2 - u_1| + m'_2 |v_2 - v_1|, \\ |f_2(t, u_2, v_2) - f_2(t, u_1, v_1)| &\leq n_1 |u_2 - u_1| + n_2 |v_2 - v_1|, \\ |g_2(t, u_2, v_2) - g_2(t, u_1, v_1)| &\leq n'_1 |u_2 - u_1| + n'_2 |v_2 - v_1|. \end{aligned}$$

with $\varpi = \max \{m_i, m'_i, n_i, n'_i\} (i = 1, 2)$.

(H2) : The functions f_1, f_2, g_1 and $g_2 : [0, 1] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ are continuous.

(H3) : There exist positive constants L_1, L_2, L'_1 and L'_2 , such that for all $t \in [0, 1], (u, v) \in \mathbb{R}^2$, we have

$$|f_i(t, u, v)| \leq L_i, i = 1, 2, \quad |g_i(t, u, v)| \leq L'_i, i = 1, 2$$

Our first main result is based on Banach contraction principle. We have:

Theorem 1. Suppose that $\gamma \neq \frac{\Gamma(p+n)}{\eta^{p+n-1}}, \delta \neq \frac{\Gamma(q+n)}{\zeta^{q+n-1}}$ and assume that (H1) holds. If

$$\varpi \max(M_1, M_2) < \frac{1}{2}, \quad (11)$$

then the fractional system (1) has a unique solution on $[0, 1]$.

Proof: Consider the operator $T : X \times Y \rightarrow X \times Y$ defined by

$$T(u, v)(t) = (T_1(u, v)(t), T_2(u, v)(t)), \quad (12)$$

where,

$$\begin{aligned} T_1(u, v)(t) = & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_1(s, u(s), v(s)) ds \\ & + \int_0^t \frac{(t-s)^{\alpha+\sigma-1}}{\Gamma(\sigma+\alpha)} g_1(s, u(s), v(s)) ds \\ & + \omega_1 t^{n-1} \int_0^\eta \frac{(\eta-s)^{p+\alpha-1}}{\Gamma(p+\alpha)} f_1(s, u(s), v(s)) ds \\ & + \omega_1 t^{n-1} \int_0^\eta \frac{(\eta-s)^{p+\alpha+\sigma-1}}{\Gamma(p+\alpha+\sigma)} g_1(s, u(s), v(s)) ds \end{aligned} \quad (13)$$

and

$$\begin{aligned} T_2(u, v)(t) = & \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f_2(s, u(s), v(s)) ds \\ & + \int_0^t \frac{(t-s)^{\theta+\beta-1}}{\Gamma(\theta+\beta)} g_2(s, u(s), v(s)) ds \\ & + \omega_2 t^{n-1} \int_0^\zeta \frac{(\zeta-s)^{q+\beta-1}}{\Gamma(q+\beta)} f_2(s, u(s), v(s)) ds \\ & + \omega_2 t^{n-1} \int_0^\zeta \frac{(\zeta-s)^{q+\beta+\theta-1}}{\Gamma(q+\beta+\theta)} g_2(s, u(s), v(s)) ds. \end{aligned} \quad (14)$$

We shall show that T is contractive:

Let $(u_1, v_1), (u_2, v_2) \in X \times Y$. Then, for each $t \in [0, 1]$, we have

$$\begin{aligned} & |T_1(u_2, v_2)(t) - T_1(u_1, v_1)(t)| \leq \\ & \leq \left(\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} ds + |\omega_1| \int_0^\eta \frac{(\eta-s)^{p+\alpha-1}}{\Gamma(p+\alpha)} ds \right) \\ & \times \sup_{0 \leq s \leq 1} |f_1(s, u_2(s), v_2(s)) - f_1(s, u_1(s), v_1(s))| \\ & + \left(\int_0^t \frac{(t-s)^{\alpha+\sigma-1}}{\Gamma(\sigma+\alpha)} ds + |\omega_1| \int_0^\eta \frac{(\eta-s)^{p+\alpha+\sigma-1}}{\Gamma(p+\alpha+\sigma)} ds \right) \\ & \times \sup_{0 \leq s \leq 1} |g_1(s, u_2(s), v_2(s)) - g_1(s, u_1(s), v_1(s))| \end{aligned} \quad (15)$$

For all $t \in [0, 1]$, and using (H1), we can write:

$$\begin{aligned} & |T_1(u_2, v_2)(t) - T_1(u_1, v_1)(t)| \\ & \leq \left(\frac{1}{\Gamma(\alpha+1)} + \frac{|\omega_1| \eta^{p+\alpha}}{\Gamma(p+\alpha+1)} \right) \\ & \times \varpi \left(\sup_{0 \leq t \leq 1} |u_2(t) - u_1(t)| + \sup_{0 \leq t \leq 1} |v_2(t) - v_1(t)| \right) \\ & + \left(\frac{1}{\Gamma(\alpha+\sigma+1)} + \frac{|\omega_1| \eta^{p+\sigma+\alpha}}{\Gamma(p+\alpha+\sigma+1)} \right) \\ & \times \varpi \left(\sup_{0 \leq t \leq 1} |u_2(t) - u_1(t)| + \sup_{0 \leq t \leq 1} |v_2(t) - v_1(t)| \right). \end{aligned} \quad (16)$$

Therefore,

$$\begin{aligned}
& |T_1(u_2, v_2)(t) - T_1(u_1, v_1)(t)| \\
& \leq \left(\frac{1}{\Gamma(\alpha+1)} + \frac{|\omega_1|\eta^{p+\alpha}}{\Gamma(p+\alpha+1)} \right) \\
& \times 2\varpi \max \{ \|u_2 - u_1\|_X, \|v_2 - v_1\|_Y \} \\
& + \left(\frac{1}{\Gamma(\alpha+\sigma+1)} + \frac{|\omega_1|\eta^{p+\sigma}}{\Gamma(p+\alpha+\sigma+1)} \right) \\
& \times 2\varpi \max \{ \|u_2 - u_1\|_X, \|v_2 - v_1\|_Y \}.
\end{aligned} \tag{17}$$

Consequently,

$$\|T_1(u_2, v_2) - T_1(u_1, v_1)\|_X \leq 2M_1\varpi \|(u_2 - u_1, v_2 - v_1)\|_{X \times Y}. \tag{18}$$

Similarly,

$$\|T_2(u_2, v_2) - T_2(u_1, v_1)\|_Y \leq 2M_2\varpi \|(u_2 - u_1, v_2 - v_1)\|_{X \times Y}. \tag{19}$$

Using (18) and (19), we deduce that

$$\begin{aligned}
& \|T(u_2, v_2) - T(u_1, v_1)\|_{X \times Y} \leq \\
& 2\varpi \max(M_1, M_2) \|(u_2 - u_1, v_2 - v_1)\|_{X \times Y}.
\end{aligned} \tag{20}$$

Thanks to (11), we conclude that T is a contraction mapping. Hence by Banach fixed point theorem, there exists a unique fixed point which is a solution of (1).

The second main result is given by the following theorem:

Theorem 2. Suppose that $\gamma \neq \frac{\Gamma(p+n)}{\eta^{p+n-1}}$, $\delta \neq \frac{\Gamma(q+n)}{\zeta^{q+n-1}}$ and assume that (H2) and (H3) are satisfied. Then the fractional system (1) has at least one solution on $[0, 1]$.

Proof: We show that the operator T is completely continuous. (Note that T is continuous on $X \times Y$ in view of the continuity of f_1, f_2, g_1 and g_2). We proceed on the following steps:

Step 1: Let us take $\bar{L} = \max \{L_i, L'_i, i = 1, 2\}$ and $B_r := \{(u, v) \in X \times Y; \|(u, v)\|_{X \times Y} \leq r, r > 0\}$. For $(u, v) \in B_r$, we have

$$\begin{aligned}
\|T_1(u, v)\|_X & \leq \frac{L_1}{\Gamma(\alpha+1)} + \frac{L'_1}{\Gamma(\alpha+\sigma+1)} \\
& + \frac{|\omega_1|L_1\eta^{p+\alpha}}{\Gamma(p+\alpha+1)} + \frac{|\omega_1|L'_1\eta^{p+\alpha+\sigma}}{\Gamma(p+\alpha+\sigma+1)},
\end{aligned} \tag{21}$$

which implies that

$$\|T_1(u, v)\|_X \leq \bar{L}M_1 < +\infty. \tag{22}$$

Similarly, it can be shown that

$$\|T_2(u, v)\|_Y \leq \bar{L}M_2 < +\infty. \quad (23)$$

By (22) and (23), we obtain

$$\|T(u, v)\|_{X \times Y} \leq \bar{L} \max\{M_1, M_2\} < +\infty. \quad (24)$$

Step 2: The equi-continuity of T :

Let $t_1, t_2 \in [0, 1]$, $t_1 < t_2$ and $(u, v) \in B_r$. We have

$$\begin{aligned} & \|T_1(u, v)(t_2) - T_1(u, v)(t_1)\|_X \leq \\ & \frac{L_1(t_2^\alpha - t_1^\alpha)}{\Gamma(\alpha+1)} + \frac{L'_1(t_2^{\alpha+\sigma} - t_1^{\alpha+\sigma})}{\Gamma(\alpha+\sigma+1)} \\ & + \left(\frac{|\omega_1|L_1\eta^{p+\alpha}}{\Gamma(p+\alpha+1)} + \frac{|\omega_1|L'_1\eta^{p+\alpha+\sigma}}{\Gamma(p+\alpha+\sigma+1)} \right) (t_2^{n-1} - t_1^{n-1}). \end{aligned} \quad (25)$$

With the same arguments, we can write

$$\begin{aligned} & \|T_2(u, v)(t_2) - T_2(u, v)(t_1)\|_Y \leq \\ & \frac{L_2(t_2^\beta - t_1^\beta)}{\Gamma(\beta+1)} + \frac{L'_2(t_2^{\theta+\beta} - t_1^{\theta+\beta})}{\Gamma(\theta+\beta+1)} \\ & + \left(\frac{|\omega_2|L_2\eta^{q+\beta}}{\Gamma(q+\beta+1)} + \frac{|\omega_2|L'_2\eta^{q+\beta+\theta}}{\Gamma(q+\beta+\theta+1)} \right) (t_2^{n-1} - t_1^{n-1}). \end{aligned} \quad (26)$$

As $t_2 \rightarrow t_1$, the right-hand sides of the inequalities (25) and (26) tend to zero. Then, as a consequence of steps 1, 2, and by Arzela-Ascoli theorem, we conclude that T is completely continuous.

Next, we show that the set

$$\Omega := \{(u, v) \in X \times Y, (u, v) = \lambda T(u, v), 0 < \lambda < 1\} \quad (27)$$

is bounded.

Let $(u, v) \in \Omega$, then $(u, v) = \lambda T(u, v)$, for some $0 < \lambda < 1$. Hence, for $t \in [0, 1]$, we have:

$$u(t) = \lambda T_1(u, v)(t), v(t) = \lambda T_2(u, v)(t). \quad (28)$$

Thanks to (H3) and using the same arguments as before, we obtain

$$\begin{aligned} \|u\|_X & \leq \lambda \|T_1(u, v)\|_X \leq \lambda \bar{L}M_1 < +\infty, \\ \|v\|_Y & \leq \lambda \|T_2(u, v)\|_Y \leq \lambda \bar{L}M_2 < +\infty. \end{aligned} \quad (29)$$

Hence, we can state that

$$\|(u, v)\|_{X \times Y} \leq \lambda \bar{L} \max \{M_1, M_2\} < +\infty. \quad (30)$$

This shows that Ω is bounded.

As a conclusion of Schaefer fixed point theorem, we deduce that T has at least one fixed point, which is a solution of (1).

The third main result is based on Krasnoselskii theorem [9]. We prove the following result:

Theorem 3. *Let $\gamma \neq \frac{\Gamma(p+n)}{\eta^{p+n-1}}$ and $\delta \neq \frac{\Gamma(q+n)}{\zeta^{q+n-1}}$. Suppose that (H1), (H2) and (H3) are satisfied. If*

$$\varpi \max \left(\frac{1}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\sigma+\alpha+1)}, \frac{1}{\Gamma(\beta+1)} + \frac{1}{\Gamma(\theta+\beta+1)} \right) < \frac{1}{2}, \quad (31)$$

then, the system (1) has at least one solution in $X \times Y$.

Proof: Let us fix $\theta \geq \max(L_1 M_1, L_2 M_2)$ and consider $B_\theta := \{(u, v) \in X \times Y, \|(u, v)\|_{X \times Y} \leq \theta\}$. On B_θ , we define the operators R and S as follows:

$$\begin{aligned} R(u, v)(t) &= (R_1(u, v)(t), R_2(u, v)(t)), \\ S(u, v)(t) &= (S_1(u, v)(t), S_2(u, v)(t)), \end{aligned} \quad (32)$$

where,

$$\begin{aligned} R_1(u, v)(t) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_1(s, u(s), v(s)) ds \\ &\quad + \int_0^t \frac{(t-s)^{\sigma+\alpha-1}}{\Gamma(\sigma+\alpha)} g_1(s, u(s), v(s)) ds, \end{aligned} \quad (33)$$

$$\begin{aligned} R_2(u, v)(t) &= \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f_2(s, u(s), v(s)) ds \\ &\quad + \int_0^t \frac{(t-s)^{\theta+\beta-1}}{\Gamma(\theta+\beta)} g_2(s, u(s), v(s)) ds, \end{aligned} \quad (34)$$

$$\begin{aligned} S_1(u, v)(t) &= \omega_1 t^{n-1} \int_0^\eta \frac{(\eta-s)^{p+\alpha-1}}{\Gamma(p+\alpha)} f_1(s, u(s), v(s)) ds \\ &\quad + \omega_1 t^{n-1} \int_0^\eta \frac{(\eta-s)^{p+\sigma-1}}{\Gamma(p+\sigma)} g_1(s, u(s), v(s)) ds \end{aligned} \quad (35)$$

and

$$\begin{aligned} S_2(u, v)(t) &= \omega_2 t^{n-1} \int_0^\zeta \frac{(\zeta-s)^{q+\beta-1}}{\Gamma(q+\beta)} f_2(s, u(s), v(s)) ds \\ &\quad + \omega_2 t^{n-1} \int_0^\zeta \frac{(\zeta-s)^{q+\theta-1}}{\Gamma(q+\theta)} g_2(s, u(s), v(s)) ds. \end{aligned} \quad (36)$$

Let $(u_1, v_1), (u_2, v_2) \in B_\theta$. Using (H3), for all $t \in [0, 1]$, we can write

$$|R_1(u_1, v_1)(t) + S_1(u_2, v_2)(t)|$$

$$\begin{aligned}
&\leq \left(\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} ds + |\omega_1| \int_0^\eta \frac{(\eta-s)^{p+\alpha-1}}{\Gamma(p+\alpha)} ds \right) \\
&\quad \times \sup_{0 \leq s \leq 1} |f_1(s, u_1(s), v_1(s))| \\
&+ \left(\int_0^t \frac{(t-s)^{\alpha+\sigma-1}}{\Gamma(\sigma+\alpha)} ds + |\omega_1| \int_0^t \frac{(t-s)^{p+\alpha+\sigma-1}}{\Gamma(p+\alpha+\sigma)} ds \right) \\
&\quad \times \sup_{0 \leq s \leq 1} |g_1(s, u_2(s), v_2(s))|.
\end{aligned} \tag{37}$$

Therefore,

$$\|R_1(u_1, v_1) + S_1(u_2, v_2)\|_X \leq L_1 M_1. \tag{38}$$

Similarly, we obtain

$$\|R_2(u_1, v_1) + S_2(u_2, v_2)\|_Y \leq L_2 M_2. \tag{39}$$

The inequalities (38) and (39) allow us to obtain

$$\|R(u_1, v_1) + S(u_2, v_2)\|_{X \times Y} \leq \max(L_1 M_1, L_2 M_2) \leq \theta. \tag{40}$$

Hence,

$$R(u_1, v_1) + S(u_2, v_2) \in B_\theta. \tag{41}$$

Now, we prove that R is contractive:

Using (H1), we can write

$$\begin{aligned}
&|R_1(u_2, v_2)(t) - R_1(u_1, v_1)(t)| \leq \\
&\frac{\varpi}{\Gamma(\alpha+1)} (\sup_{0 \leq s \leq 1} |u_2(s) - u_1(s)| + \sup_{0 \leq s \leq 1} |v_2(s) - v_1(s)|) \\
&+ \frac{\varpi}{\Gamma(\sigma+\alpha+1)} (\sup_{0 \leq s \leq 1} |u_2(s) - u_1(s)| + \sup_{0 \leq s \leq 1} |v_2(s) - v_1(s)|).
\end{aligned} \tag{42}$$

Thus,

$$\begin{aligned}
&\|R_1(u_2, v_2) - R_1(u_1, v_1)\|_X \leq \\
&\left(\frac{2\varpi}{\Gamma(\alpha+1)} + \frac{2\varpi}{\Gamma(\sigma+\alpha+1)} \right) \|(u_2 - u_1, v_2 - v_1)\|_{X \times Y}.
\end{aligned} \tag{43}$$

Similarly,

$$\begin{aligned}
&\|R_2(u_2, v_2) - R_2(u_1, v_1)\|_Y \leq \\
&\left(\frac{2\varpi}{\Gamma(\beta+1)} + \frac{2\varpi}{\Gamma(\theta+\beta+1)} \right) \|(u_2 - u_1, v_2 - v_1)\|_{X \times Y}.
\end{aligned} \tag{44}$$

Consequently,

$$\begin{aligned}
&\|R(u_2, v_2) - R(u_1, v_1)\|_{X \times Y} \leq \\
&2\varpi \max \left(\frac{1}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\sigma+\alpha+1)}, \frac{1}{\Gamma(\beta+1)} + \frac{1}{\Gamma(\theta+\beta+1)} \right) \\
&\quad \times \|(u_2 - u_1, v_2 - v_1)\|_{X \times Y}.
\end{aligned} \tag{45}$$

Thanks to (31), we conclude that R is a contraction mapping.

The continuity of f_1, f_2, g_1 and g_2 given in (H2) implies that the operator S is continuous.

Now, we prove the compactness of the operator S :

Let $t_1, t_2 \in [0, 1], t_1 < t_2$ and $(u, v) \in B_\theta$. By (H3), we have

$$\begin{aligned} & \|S_1(u, v)(t_2) - S_1(u, v)(t_1)\|_X \leq \\ & |\omega_1| \left(\frac{L_1 \eta^{p+\alpha}}{\Gamma(p+\alpha+1)} + \frac{L'_1 \eta^{p+\alpha+\sigma}}{\Gamma(p+\alpha+\sigma+1)} \right) (t_2^{n-1} - t_1^{n-1}). \end{aligned} \quad (46)$$

With same arguments as before, we have

$$\begin{aligned} & \|S_2(u, v)(t_2) - S_2(u, v)(t_1)\|_Y \leq \\ & |\omega_2| \left(\frac{L_2 \xi^{q+\beta}}{\Gamma(q+\beta+1)} + \frac{L'_2 \xi^{q+\beta+\theta}}{\Gamma(q+\beta+\theta+1)} \right) (t_2^{n-1} - t_1^{n-1}). \end{aligned} \quad (47)$$

The right hand sides of (46) and (47) are independent of (u, v) and tend to zero as $t_1 \rightarrow t_2$, so S is relatively compact on B_θ . Then by Ascoli-Arzelà theorem, the operator S is compact. Finally, by Krasnoselskii theorem, we conclude that there exists a solution to (1). Theorem 3 is thus proved.

4 Examples

In this section, we present some examples to illustrate the main results.

Example 1. Consider the following fractional differential system:

$$\left\{ \begin{array}{l} D^{\frac{5}{2}} u(t) = \frac{\sin(u(t)+v(t))}{32(t^2+1)} + \int_0^t \frac{(t-s)^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})} \frac{\sin u(s)+\sin v(s)}{16(t+1)} ds + 2t + 1, \\ D^{\frac{5}{2}} v(t) = \frac{|u(t)|+|v(t)|}{(t^2+t+20)(1+|u(t)|+|v(t)|)} + \exp(2t) \\ + \int_0^t \frac{(t-s)^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})} \left(\frac{|u(s)|}{(15+t)(1+|u(s)|)} + \frac{|v(s)|}{(t+10)(1+|v(s)|)} \right) ds, \\ |u(0)| + |v(0)| + |u'(0)| + |v'(0)| = 0, \\ u''(0) = \gamma I^p u(\eta), \eta \in]0, 1[, \\ v''(0) = \delta I^q v(\zeta), \zeta \in]0, 1[, \end{array} \right. \quad (48)$$

where, $\alpha = \beta = \frac{5}{2}, \sigma = \theta = \frac{1}{2}, \eta = \zeta = \frac{1}{2}, p = q = \frac{3}{2}, \gamma = -4, \delta = -8$, and for all $(u, v) \in \mathbb{R}^2, t \in [0, 1]$, we have

$$\begin{aligned} f_1(t, u, v) &= \frac{\sin(u+v)}{32(t^2+1)} + 2t + 1, \\ g_1(t, u, v) &= \frac{\sin u + \sin v}{16(t+1)}, \\ f_2(t, u, v) &= \frac{|u| + |v|}{(t^2 + t + 20)(1 + |u| + |v|)} + \exp(2t), \\ g_2(t, u, v) &= \frac{|u|}{(15+t)(1+|u|)} + \frac{|v|}{(t+10)(1+|v|)}. \end{aligned}$$

We have also

$$\gamma \neq \frac{\Gamma(p+n)}{\eta^{p+n-1}} = 105\sqrt{2\pi}, \delta \neq \frac{\Gamma(q+n)}{\zeta^{q+n-1}} = 105\sqrt{2\pi}.$$

For $(u_1, v_1), (u_2, v_2) \in \mathbb{R}^2, t \in [0, 1]$, we have

$$\begin{aligned} |f_1(t, u_2, v_2) - f_1(t, u_1, v_1)| &\leq \frac{1}{32} (|u_2 - u_1| + |v_2 - v_1|), \\ |g_1(t, u_2, v_2) - g_1(t, u_1, v_1)| &\leq \frac{1}{16} (|u_2 - u_1| + |v_2 - v_1|), \\ |f_2(t, u_2, v_2) - f_2(t, u_1, v_1)| &\leq \frac{1}{20} (|u_2 - u_1| + |v_2 - v_1|), \\ |g_2(t, u_2, v_2) - g_2(t, u_1, v_1)| &\leq \frac{1}{15} (|u_2 - u_1| + |v_2 - v_1|). \end{aligned}$$

It yields then that

$$M_1 = 0.774, M_2 = 0.788, \varpi = \frac{1}{10}, 2\varpi \max(M_1, M_2) = 0.157 < 1.$$

The conditions of Theorem 1 hold. Therefore, the problem (48) has a unique solution on $[0, 1]$.

We give also the following example:

Example 2. We take:

$$\left\{ \begin{array}{l} D^{\frac{13}{4}} u(t) = \exp(-t^2) (\sin(u(t)) + \cos(v(t))) \\ \quad + \int_0^t \frac{(t-s)^{\frac{5}{2}}}{\Gamma(\frac{7}{2})} \sin(t+u(s)v(s)) ds, t \in [0, 1], \\ D^{\frac{22}{7}} v(t) = \exp(t^2) (\sin(u(t)) + v(t)) \\ \quad + \int_0^t \frac{(t-s)^{\frac{5}{2}}}{\Gamma(\frac{7}{2})} \exp(-t|u(s)| - |v(s)|) ds, t \in [0, 1], \\ u(0) = u'(0) = v(0) = v'(0) = 0. \end{array} \right. \quad (49)$$

For this problem, we remark that $\alpha = \frac{13}{4}, \beta = \frac{22}{7}, p = 3, q = 2, \gamma = \delta = \sqrt{2}, \eta = \frac{4}{5}, \xi = \frac{1}{5}$, and for all $(u, v, z) \in \mathbb{R}^3$, we have

$$\begin{aligned} f_1(t, u, v) &= \exp(-t^2) (\sin u + \cos v), \\ g_1(t, u, v) &= (\sin(t + uv)), \\ f_2(t, u, v) &= \exp(t^2) (\sin u + v), \\ g_2(t, u, v) &= \exp(-t|u| - |v|). \end{aligned}$$

It's clear that the conditions of Theorem 2 are satisfied. Then the problem (49) has at least one solution on $[0, 1]$.

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